GROUP THEORY 2024 - 25, SOLUTION SHEET 6

Exercise 1. To do yourself. Ask the assistant if something is unclear.

Exercise 2. (1) For order 180:

- Factor 180:
$$180 = 2^2 \cdot 3^2 \cdot 5.$$
- By the classification theorem for finitely generated abelian groups, any abelian group G of order 180 decomposes as a direct sum of cyclic groups corresponding to these prime powers.
  - For the 2-part (order $2^2 = 4$), the possible cyclic groups are $\mathbb{Z}_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$.
  - Possible structures for the 3-part (order $3^2 = 9$) are:
    * $\mathbb{Z}_9$
    * $\mathbb{Z}_3 \times \mathbb{Z}_3$
  - For the 5-part (order 5), the only option is $\mathbb{Z}_5$.
- Therefore, the possible classifications for G are:
$$G \cong \mathbb{Z}_4 \times \mathbb{Z}_9 \times \mathbb{Z}_5 \quad , \quad G \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_9 \times \mathbb{Z}_5$$
$$G \cong \mathbb{Z}_4 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_5 \quad , \quad G \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_5$$

(2) For order 72:

- Factor 72:
$$72 = 2^3 \cdot 3^2.$$
- Possible structures for the 2-part (order $2^3 = 8$) are:
  - $\mathbb{Z}_8$
  - $\mathbb{Z}_4 \times \mathbb{Z}_2$
  - $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$
- Possible structures for the 3-part (order $3^2 = 9$) are:
  - $\mathbb{Z}_9$
  - $\mathbb{Z}_3 \times \mathbb{Z}_3$
- By combining these, the possible classifications for G are:
$$G \cong \mathbb{Z}_8 \times \mathbb{Z}_9, \quad \mathbb{Z}_8 \times \mathbb{Z}_3 \times \mathbb{Z}_3, \quad \mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_9,$$
$$\mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3, \quad \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_9, \quad \text{and} \quad \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3.$$

(3) For order 200:

- Factor 200:
$$200 = 2^3 \cdot 5^2.$$
- Possible structures for the 2-part (order $2^3 = 8$) are:
  - $\mathbb{Z}_8$
  - $\mathbb{Z}_4 \times \mathbb{Z}_2$
  - $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$

- Possible structures for the 5-part (order $5^2 = 25$) are:
  - $\mathbb{Z}_{25}$
  - $\mathbb{Z}_5 \times \mathbb{Z}_5$
- By combining these, the possible classifications for G are:

$$G \cong \mathbb{Z}_8 \times \mathbb{Z}_{25}, \quad \mathbb{Z}_8 \times \mathbb{Z}_5 \times \mathbb{Z}_5, \quad \mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_{25},$$
$$\mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_5 \times \mathbb{Z}_5, \quad \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{25}, \quad \text{and} \quad \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_5 \times \mathbb{Z}_5.$$

Exercise 3. *Classification of finite abelian groups*

(1) We have $100 = 2^2 \cdot 5^2$. If A has no element of order 4, then A cannot have a subgroup that is isomorphic to $\mathbb{Z}/4\mathbb{Z}$, so the latter cannot appear in the decomposition of A given by the classification theorem of finite abelian groups. Thus, A must be isomorphic to one of the following groups:

$$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/25\mathbb{Z} \text{ or } \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times Z/5\mathbb{Z} \times Z/5\mathbb{Z}$$

In particular, A has a subgroup isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

(2) By the classification theorem of finite ableian groups, the abelian groups of order p^5 are, up to isomorphism, the following:

$$\mathbb{Z}/p^5\mathbb{Z}, \quad \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p^4\mathbb{Z}, \quad \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p^3\mathbb{Z}$$
$$\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p^2\mathbb{Z}, \quad \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$$
$$\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p^2\mathbb{Z} \times \mathbb{Z}/p^2\mathbb{Z}, \quad \mathbb{Z}/p^2\mathbb{Z} \times \mathbb{Z}/p^3\mathbb{Z}$$

Thus, there are exactly 7 such groups. Each one of these corresponds to a partition of the integer 5, i.e. the number of different ways to write n as a sum of positive integers. By the same theorem, we can check that the number of abelian groups of order p^n corresponds to the number of partitions of the integer n.

Exercise 4. (1) Note that since each $\mathrm{Tors}(A_\alpha)$ is a subgroup of A_α, we have that $\bigoplus_{\alpha\in I} \mathrm{Tors}(A_\alpha)$ is a subgroup of $\bigoplus_{\alpha\in I} A_\alpha$ and by definition so is $\mathrm{Tors}(\bigoplus_{\alpha\in I} A_\alpha)$. We show that they are the same set. Let $(a_\alpha)_{\alpha\in I} \in \mathrm{Tors}(\bigoplus_{\alpha\in I} A_\alpha)$. So there exists $n > 0$ such that $n(a_\alpha)_{\alpha\in I} = 0$. Therefore for all $\alpha \in I$ we obtain that $na_\alpha = 0$ and consequently $a_\alpha \in \mathrm{Tors}(A_\alpha)$. Hence $(a_\alpha)_{\alpha\in I} \in \bigoplus_{\alpha\in I} \mathrm{Tors}(A_\alpha)$ and $\mathrm{Tors}(\bigoplus_{\alpha\in I} A_\alpha) \subseteq \bigoplus_{\alpha\in I} \mathrm{Tors}(A_\alpha)$.

Conversely if $(a_\alpha)_{\alpha\in I} \in \bigoplus_{\alpha\in I} \mathrm{Tors}(A_\alpha)$, then for each $\alpha \in I$ let n_α be the minimum positive integer such that $n_\alpha a_\alpha = 0$. Since all but finitely many a_α are 0, all but finitely many $n_\alpha = 1$ and so we can define $n = \Pi_\alpha n_\alpha$. Since $n((a_\alpha)_{\alpha\in I}) = 0$, we obtain that $(a_\alpha)_{\alpha\in I} \in \mathrm{Tors}(\bigoplus_{\alpha\in I} A_\alpha)$.

(2) The proof of the first inclusion for direct sums goes through in the case of direct products.

Let $A = \prod_{n>1} \mathbb{Z}/n\mathbb{Z}$. Then $(1, 1, 1, ...) \in \prod_{n>1} \mathrm{Tors}(\mathbb{Z}/n\mathbb{Z})$ but one checks that $(1, 1, 1...) \notin \mathrm{Tors}(\prod_{n>1} \mathbb{Z}/n\mathbb{Z})$.

Exercise 5. Define the homomorphism $\phi : \mathbb{Z} \to \mathbb{Z}/p_1^{a_1}\mathbb{Z} \times \mathbb{Z}/p_2^{a_2}\mathbb{Z} \times \cdots \times \mathbb{Z}/p_k^{a_k}\mathbb{Z}$ by

$$\phi(x) = \left(x \bmod p_1^{a_1}, x \bmod p_2^{a_2}, \ldots, x \bmod p_k^{a_k}\right),$$

which maps each integer x to its equivalence classes modulo $p_1^{a_1}, p_2^{a_2}, \ldots, p_k^{a_k}$. By the First Isomorphism Theorem

$$\mathbb{Z}/\ker(\phi) \cong \operatorname{im}(\phi).$$

The kernel consists of all integers x such that

$$\phi(x) = (0, 0, \ldots, 0).$$

This means that $x \equiv 0 \pmod{p_i^{a_i}}$ for each $i = 1, 2, \ldots, k$. Therefore, x must be a multiple of $d = p_1^{a_1} p_2^{a_2} \ldots p_k^{a_k}$, as this is the smallest integer divisible by each $p_i^{a_i}$. Thus, $\ker(\phi) = d\mathbb{Z}$.

To show that ϕ is surjective, consider an arbitrary element $(y_1, y_2, \ldots, y_k)$ in

$$\mathbb{Z}/p_1^{a_1}\mathbb{Z} \times \mathbb{Z}/p_2^{a_2}\mathbb{Z} \times \cdots \times \mathbb{Z}/p_k^{a_k}\mathbb{Z}.$$

We need to find an integer $x \in \mathbb{Z}$ such that

$$x \equiv y_i \pmod{p_i^{a_i}} \quad \text{for each } i = 1, 2, \ldots, k.$$

For each i, define

$$p_i' = \frac{d}{p_i^{a_i}} = p_1^{a_1} \ldots p_{i-1}^{a_{i-1}} p_{i+1}^{a_{i+1}} \ldots p_k^{a_k},$$

which is coprime to $p_i^{a_i}$. By Bezout's Lemma, there exists an integer b_i such that

$$p_i' b_i \equiv 1 \pmod{p_i^{a_i}}. \tag{1}$$

Now define

$$x = y_1 p_1' b_1 + y_2 p_2' b_2 + \cdots + y_k p_k' b_k.$$

This element x satisfies $\phi(x) = (y_1, \ldots, y_k)$ using (1) and the fact that $p_i' = 0 \pmod{p_j^{a_j}}$ for all $i \neq j$.

Therefore, ϕ is surjective which concludes the proof.

Exercise 6. *Divisible abelian groups*

(1) $(\mathbb{Q}, +)$ is such an example.
(2) Consider the group $\mathbb{Z}/3\mathbb{Z}$. Clearly, as $3x = 0$ for all $x \in \mathbb{Z}/3\mathbb{Z}$, this group is not 3-divisible. However, as $2 \cdot 1 = 2, 2 \cdot 2 = 1$, we can see that it is 2-divisible.
(3) We give two examples:
  - The product $\mathbb{Q} \times \mathbb{Z}/3\mathbb{Z}$ is clearly infinite, 2-divisible, but not 3-divisible.
  - Consider the (additive) group $\mathbb{Z}_2 := \{\frac{a}{2^i} | a, i \in \mathbb{Z}, 2 \nmid a\} \bigcap \{0\} \subseteq \mathbb{Q}$. Firstly, let us prove that this is indeed a (sub)group (of $\mathbb{Q}$). By definition, it clearly contains the neutral element and all inverses of its elements. Let us verify that it is stable by addition: for $i < j$:

$$\frac{a}{2^i} + \frac{b}{2^j} = \frac{a2^{j-i} + b}{2_j} \in \mathbb{Z}_2$$

because $a2^{j-i} + b$ is odd. If $i = j$ and $a = -b$, then

$$\frac{a}{2^i} + \frac{b}{2^j} = 0 \in \mathbb{Z}_2$$

If $i = j$ and $a \neq -b$, write $a + b = c2^k$ with $2 \nmid c$ and $k \geq 1$. We then have

$$\frac{a}{2^i} + \frac{b}{2^j} = \frac{a+b}{2^j} = \frac{c}{2^{i-k}} \in \mathbb{Z}_2$$

so the latter is indeed a group.
Observe that $\mathbb{Z}_2$ is not 3-divisible because $\frac{1}{3} \notin \mathbb{Z}_2$ and thus there is no element $x \in \mathbb{Z}_2$ such that $3x = 1 \in \mathbb{Z}_2$. 2-divisibility is straightforward.

(4) We give two proofs:
- Let $n = |G|$ be the cardinal of G and let $g \in G$. Since g is n-divisible there exists $g_0 \in G$ such that $g = ng_0$. But for every element we have that $ng_0 = 0$ (since the order $o(g_0)$ divides n, $ng_0 = ko(g_0)g_0 = 0$ for some $k \in \mathbb{N}$).
- Suppose by contradiction that G is finite, divisible, non-trivial and let $0 \neq g \in G$. Denote $n := |G| = p_1^{n_1} \dots p_m^{n_m}$ with $p_1, \dots, p_m$ distinct primes and all $n_i \geq 1$. By inductively applying p_i-divisibility, there exists g_1 such that $p_1^{n_1} g_1 = g$ and g_i for $i = 2, \dots, m$ such that $p_i^{n_i} g_i = g_{i-1}$. In particular, we have $0 = ng_m = g \neq 0$, which is absurd.

Exercise 7. We need to find finitely generated abelian groups G upto isomorphism which fit into the exact sequence:

$$0 \to \mathbb{Z} \xrightarrow{i} G \xrightarrow{f} \mathbb{Z}/12\mathbb{Z} \to 0.$$

We claim that such G are given up to isomorphism by $\mathbb{Z} \times \mathbb{Z}/d\mathbb{Z}$ for $d \mid 12$. Since G is finitely generated, it is isomorphic to $F \times T$ where F is a free group isomorphic to $\mathbb{Z}^l$ for some $l \geq 0$ and T is a torsion group.

Note that $\operatorname{Ker} f = \operatorname{Im} i \cong \mathbb{Z}$ and therefore has no torsion. So we obtain that $f_{|T}$ is injective. Hence T is isomorphic to a subgroup of $\mathbb{Z}/12\mathbb{Z}$. Therefore

$$T \cong \mathbb{Z}/d\mathbb{Z}.$$

where $d \mid 12$.

If we restrict the short exact exact sequence to the free group $F \times \{0\}$, we obtain that F surjects onto a finite abelian group and has a kernel isomorphic to $\mathbb{Z}$. We let the reader convince themselves that this implies that the rank of the free Abelian group F is 1. Hence we obtain that $G \cong \mathbb{Z} \times \mathbb{Z}/d\mathbb{Z}$ for some $d \mid 12$.

It remains to show that for each $d \mid 12$ there is an exact sequence of the form

$$0 \to \mathbb{Z} \xrightarrow{i} \mathbb{Z} \times \mathbb{Z}/d\mathbb{Z} \xrightarrow{f} \mathbb{Z}/12\mathbb{Z} \to 0.$$

To this end let $d' = \frac{12}{d}$ and let $f : \mathbb{Z} \times \mathbb{Z}/d\mathbb{Z} \to \mathbb{Z}/12\mathbb{Z}$, $(a, \bar{b}) \mapsto a + d'b + 12\mathbb{Z}$. Note that f is a surjection and it's kernel has no non-trivial torsion elements. Since $\operatorname{Ker} f$ has no torsion and $\operatorname{Ker} f \subseteq \mathbb{Z} \times \mathbb{Z}/d\mathbb{Z}$, it follows from the classification theorem of finitely generated Abelian groups that $\operatorname{Ker} f \cong \mathbb{Z}$. Hence we obtain an exact sequence as above. $\square$

Exercise 8. (1) Let $x \in G$ such that $x \neq e$. By Lagrange, the order of x divides the order of G, so the order of the former is p^k for some $1 \leq k \leq n$. Then we can check that $x^{p^{k-1}}$ is an element of order p.

(2) Let us proceed by induction on k. For $k = 0$, $\{e\}$ is a normal subgroup of order $p^0 = 1$ of G. If $k = n$, G itself is a normal subgroup of order p^n. Suppose $0 < k < n$ and that there exists a normal subgroup N of G of order p^{k-1}. Then G/N is a nontrivial p-group. By exercise 2 of sheet 4 (the same proof works) and Lagrange, $\mathrm{Z}(G/N) \neq 0$ and thus contains an element xN of order p. Consider the quotient homomorphism $\pi : G \to G/N$ and let us prove that $\pi^{-1}(\langle xN \rangle)$ is a normal subgroup of order p^k of G. These follow from the following observations: as $\langle xN \rangle < \mathrm{Z}(G/N)$, $\langle xN \rangle$ is a normal subgroup of G/N and thus the preimage by π is a normal subgroup of G. Moreover, by the first isomorphism theorem, we find $|\pi^{-1}(\langle xN \rangle)| = |\langle xN \rangle| \cdot |\mathrm{Ker}(\pi)| = p^k$.

(3) We construct the desired chain by induction, where $G_k < G_{k-1}$ is normal for all $1 \leq k \leq n$ and $|G_K| = p^{n-k}$ for all $0 \leq k \leq n$. Let $G_0 = G$ and $k > 0$. By induction, we bave G_{k-1} of order p^{n-k+1}. By (ii) there exists a normal subgroup $G_k < G_{k-1}$ of order p^{n-k}. Moreover, for all k the order of G_k/G_{k-1} is equal to p, so all the quotients are cyclic, thus abelian, CQFD.